

Vectorial $\text{AdS}_5/\text{CFT}_4$ duality for spin-one boundary theory

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Abstract

We consider an example of vectorial $\text{AdS}_5/\text{CFT}_4$ duality when the boundary theory is described by free N complex or real Maxwell fields. It is dual to a particular (“type C”) higher spin theory in AdS_5 containing fields in special mixed-symmetry representations. We extend the study of this theory in arXiv:1410.3273 by deriving the expression for the large N limit of the corresponding singlet-sector partition function on $S^1 \times S^3$. We find that in both complex $U(N)$ and real $O(N)$ invariant cases the form of the one-particle partition function is as required by the AdS/CFT duality. We also discuss the matching of the Casimir energy on S^3 by assuming an integer shift in the bulk theory coupling.

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1 Introduction

In addition to “adjoint” AdS/CFT duality between gauge theory at the boundary and string theory in the bulk there is a simpler example of “vectorial” AdS/CFT relating singlet sector of conserved currents of a CFT described by N free massless fields in a vector representation of $U(N)$ or $O(N)$ to a Vasiliev-type theory in AdS. This duality is not restricted to original examples in $d = 3$ [1, 2]: generalizations to $d > 3$ were studied, e.g., in [3, 4, 5, 6, 7, 8].¹ While in $d = 3$ the only option to get a unitary theory with a higher spin symmetry is to use free scalars or spin $\frac{1}{2}$ fermions [10] in the $d = 4$ case that we will be interested in here one is also allowed to consider free spin 1 fields [11, 12, 13].²

The corresponding conserved currents that appear in the product of two spin 1 doubletons (as in [16, 17, 18, 19, 20, 21]) are in specific mixed-symmetry representations of $SO(2, 4)$.³ The singlet sector of a theory of N complex or real Maxwell vectors invariant under $U(N)$ or $O(N)$ should then be dual to a particular version of higher spin theory in AdS_5 involving mixed-symmetry fields. This theory was called “type C” in [25] by analogy with type A theory dual to N boundary scalars and type B theory dual to N spin $\frac{1}{2}$ boundary fermions. Here we will elaborate on its discussion in [25] by directly computing the singlet-sector partition in this theory and explaining the matching of the AdS_5 vacuum energy in the bulk and the S^3 Casimir energy at the boundary.

2 Representation content and relations between characters

Let us first review the case when the boundary theory is described by N complex or real scalars. We shall use the notation $(\Delta; j_1, j_2)$ for generic “massive” representation of $SO(2, 4)$, with the “massless” case corresponding to $\Delta = 2 + j_1 + j_2$ with $j_1 j_2 > 0$.⁴ Also, $\{j, 0\}$ and $\{0, j\}$ with $\Delta = 1 + j$ will denote spin j doubleton representation. The spectrum of states in “non-minimal” type A theory dual to complex $U(N)$ scalar theory may be found from the decomposition of the product of two $j = 0$ doubletons [19] (generalizing the $d = 3$ relation of [26])

$$\text{non - minimal A :} \quad \{0, 0\} \otimes \{0, 0\} = (2; 0, 0) + \bigoplus_{s=1}^{\infty} (2 + s; \frac{s}{2}, \frac{s}{2}) . \quad (2.1)$$

Here the representations $(2 + s; \frac{s}{2}, \frac{s}{2})$ correspond to conserved currents dual to massless totally symmetric spin s fields in AdS_5 . In the “minimal” type A theory case dual to the real scalar $O(N)$

¹An early discussion of $d = 4$ case based on gauged $O(N)$ model appeared in [9].

²Higher-spin symmetries of such boundary models and their higher-spin and super-extensions were found in [14]. More generally, one may also attempt to use a higher-spin doubletons at the boundary; this possibility was noticed in [15] where the corresponding higher spin algebras were studied.

³Mixed-symmetry fields in AdS_5 and the associated currents were discussed, e.g., in [22, 23, 24].

⁴Here (j_1, j_2) are $SU(2) \times SU(2)$ weights (with total spin being $s = j_1 + j_2$) and we shall use the notation $(\Delta; j_1, j_2)_c = (\Delta; j_1, j_2) + (\Delta; j_2, j_1)$.

theory one is to project out all odd spin fields (the corresponding currents become trivial), i.e. to “symmetrize” the product

$$\text{minimal A : } [\{0, 0\} \otimes \{0, 0\}]_{\text{sym}} = (2; 0, 0) + \bigoplus_{s=2,4,\dots}^{\infty} (2 + s; \frac{s}{2}, \frac{s}{2}) . \quad (2.2)$$

Let us define the (“blind”) characters for the basic representations ($\Delta_0 = 2 + j_1 + j_2$) [20]⁵

$$\text{“massive” : } Z(\Delta; j_1, j_2) = \frac{q^\Delta}{(1-q)^4} (2j_1 + 1) (2j_2 + 1) , \quad (2.3)$$

$$\text{“massless” : } Z(\Delta_0; j_1, j_2) = \frac{q^{\Delta_0}}{(1-q)^4} [(2j_1 + 1)(2j_2 + 1) - 4q j_1 j_2] , \quad (2.4)$$

$$\text{“doubleton” : } Z(\{j, 0\}) = Z(\{0, j\}) = \frac{q^{1+j}}{(1-q)^3} [2j + 1 - q(2j - 1)] . \quad (2.5)$$

Here (2.3)/(2.4) has the interpretation of one-particle partition function $Z(\Delta_0; j_1, j_2)$ for the corresponding massive/massless 5d field in AdS_5 with standard boundary conditions. The character identities which are the counterparts of (2.1) and (2.2) are [20]

$$\text{non - minimal A : } [Z(\{0, 0\})]^2 = Z(2; 0, 0) + \sum_{s=1}^{\infty} Z(2 + s; \frac{s}{2}, \frac{s}{2}) , \quad (2.6)$$

and [8]

$$\text{minimal A : } \frac{1}{2} [Z(\{0, 0\})]^2 + \frac{1}{2} [Z(\{0, 0\})]_{q \rightarrow q^2} = Z(2; 0, 0) + \sum_{s=2,4,\dots}^{\infty} Z(2 + s; \frac{s}{2}, \frac{s}{2}) . \quad (2.7)$$

The validity of (2.7) can be verified directly, but the group-theoretic origin of the combination in the l.h.s. is not obvious. The l.h.s. parts of (2.6) and (2.7) have the interpretation of the large N limit of singlet-sector one-particle partition functions of the $U(N)$ [27] and the $O(N)$ [8] scalar theories. The r.h.s. parts are interpreted as the total contribution to the one-particle partition function of corresponding higher spin theory defined on thermal quotient of AdS_5 [28, 29].

The same construction can be repeated for spin $\frac{1}{2}$ boundary theory when the role of $j = 0$ doubleton is played by the $j = \frac{1}{2}$ one, leading to the spectrum of type B higher spin theory in AdS_5 [8, 25]. Similarly, taking the product of two $j = 1$ doubletons $\{1, 0\}_c = \{1, 0\} + \{0, 1\}$ (that may be associated with the self-dual and anti self-dual parts of Maxwell field strength F_{mn}) we get the spectrum of conserved currents and other primary fields and thus the content of the corresponding type C theory

⁵We follow the notation of [25]. Eq.(2.3) applies also to the case of massive selfdual representations that appear when $j_1 j_2 = 0$ and $\Delta > 1 + j_1 + j_2$.

in AdS₅ [25]. In general, the products of two doubleton representations decompose as follows [19, 20]

$$\{j, 0\} \otimes \{j', 0\} = \bigoplus_{k=|j-j'|}^{j+j'} (2+j+j'; k, 0) + \bigoplus_{k=1}^{\infty} (2+j+j'+k; j+j'+\frac{k}{2}, \frac{k}{2}) , \quad (2.8)$$

$$\{0, j\} \otimes \{0, j'\} = \bigoplus_{k=|j-j'|}^{j+j'} (2+j+j'; 0, k) + \bigoplus_{k=1}^{\infty} (2+j+j'+k; \frac{k}{2}, j+j'+\frac{k}{2}) , \quad (2.9)$$

$$\{j, 0\} \otimes \{0, j'\} = \bigoplus_{k=0}^{\infty} (2+j+j'+k; j+\frac{k}{2}, j'+\frac{k}{2}) . \quad (2.10)$$

For an “unsymmetrized” product of two spin 1 doubletons one then finds the spectrum of the non-minimal type C theory dual to the singlet sector of N complex Maxwell fields at the boundary

$$\begin{aligned} \text{non - minimal C :} \quad & \{1, 0\}_c \otimes \{1, 0\}_c \equiv (\{1, 0\} + \{0, 1\}) \otimes (\{1, 0\} + \{0, 1\}) \\ & = 2(4; 0, 0) + (4; 1, 0)_c + (4; 2, 0)_c + 2 \bigoplus_{k=0}^{\infty} (4+k; \frac{k+2}{2}, \frac{k+2}{2}) + \bigoplus_{k=1}^{\infty} (4+k; 2+\frac{k}{2}, \frac{k}{2})_c \\ & = 2(4; 0, 0) + (4; 1, 0)_c + 2 \bigoplus_{s=2}^{\infty} (2+s; \frac{s}{2}, \frac{s}{2}) + \bigoplus_{s=2}^{\infty} (2+s; \frac{s+2}{2}, \frac{s-2}{2})_c . \end{aligned} \quad (2.11)$$

In addition to two infinite series of massless spin $s \geq 2$ fields it contains also a massive scalar and pseudoscalar in representation $(4; 0, 0)$ (dual to $F_{mn}^* F^{mn}$ and $F_{mn}^* \tilde{F}^{mn}$) and the rank 2 antisymmetric tensor in self-dual and anti self-dual massive representation $(4; 1, 0)_c$ (dual to $F_{m[n}^* F_{k]m}$). The spectrum of minimal type C theory dual to real $O(N)$ Maxwell theory is found by projecting out one parity-odd symmetric tensor states and odd-spin mixed-symmetry states:

$$\begin{aligned} \text{minimal C :} \quad & [\{1, 0\}_c \otimes \{1, 0\}_c]_{\text{sym}} \\ & = 2(4; 0, 0) + \bigoplus_{s=2}^{\infty} (2+s; \frac{s}{2}, \frac{s}{2}) + \bigoplus_{s=2,4,\dots}^{\infty} (2+s; \frac{s+2}{2}, \frac{s-2}{2})_c . \end{aligned} \quad (2.12)$$

The corresponding character relations are counterparts of (2.6) and (2.7) in type A case:

$$\begin{aligned} \text{non - minimal C :} \quad & [Z(\{1, 0\}_c)]^2 \\ & = 2Z(4; 0, 0) + Z(4; 1, 0)_c + 2 \sum_{s=2}^{\infty} Z(2+s; \frac{s}{2}, \frac{s}{2}) + \sum_{s=2}^{\infty} Z(2+s; \frac{s+2}{2}, \frac{s-2}{2})_c , \end{aligned} \quad (2.13)$$

$$\begin{aligned} \text{minimal C :} \quad & \frac{1}{2} [Z(\{1, 0\}_c)]^2 + \frac{1}{2} [Z(\{1, 0\}_c)]_{q \rightarrow q^2} \\ & = 2Z(4; 0, 0) + \sum_{s=2}^{\infty} Z(2+s; \frac{s}{2}, \frac{s}{2}) + \sum_{s=2,4,\dots}^{\infty} Z(2+s; \frac{s+2}{2}, \frac{s-2}{2})_c . \end{aligned} \quad (2.14)$$

Like in the type A case and the type B cases [8] these relations between characters have again a field theory or AdS/CFT interpretation. As we shall show below, the l.h.s. of (2.13) is the one-particle

partition function representing the leading term in the large N limit of the singlet-sector partition function of the theory of N complex Maxwell fields, while the l.h.s. of (2.14) corresponds to the real $O(N)$ Maxwell theory case.

3 Singlet-sector partition function in the theory of N Maxwell fields

Let us first recall the expression for the partition function of one scalar field on $S^1 \times S^3$, then consider N scalars and impose the singlet-sector constraint and, finally, generalize to the case of Maxwell vectors instead of scalars.

For a conformal scalar we get the partition function (we assume S^3 to have unit radius and length of S^1 to be β)

$$Z_0 = (\det \mathcal{O}_0)^{-1/2}, \quad \mathcal{O}_0 = -D^2 + \frac{1}{6}R = -\partial_0^2 - \mathbf{D}^2 + 1. \quad (3.1)$$

Using the eigenvalues of the Laplacian $-\mathbf{D}^2$ on S^3 we get the eigenvalues and multiplicities of \mathcal{O}_0 ($k = 0, \pm 1, \dots$, $n = 0, 1, 2, \dots$)

$$\lambda_{k,n} = w_k^2 + \omega_n^2, \quad w_k = 2\pi k\beta^{-1}, \quad \omega_n = n + 1, \quad d_n = (n + 1)(n + 2). \quad (3.2)$$

Thus $\log Z_0 = -\frac{1}{2} \log \det \mathcal{O}_0 = -\frac{1}{2} \sum_{k,n} d_n \log \lambda_{k,n}$.

In general, for the bosonic partition function we have ($q \equiv e^{-\beta}$)

$$\Gamma = -\log Z = \beta E_c + F(\beta), \quad E_c = \frac{1}{2} \sum_n d_n \omega_n, \quad (3.3)$$

$$F = \sum_n d_n \log(1 - e^{-\beta \omega_n}) = - \sum_{m=1}^{\infty} \frac{1}{m} \mathcal{Z}(q^m), \quad \mathcal{Z}(q) = \sum_n d_n e^{-\beta \omega_n}, \quad (3.4)$$

where E_c is Casimir energy on S^3 and \mathcal{Z} is one-particle partition function. For the massless 4d scalar \mathcal{Z} thus has the same expression as the character for spin 0 doubleton in (2.5)

$$\mathcal{Z}_0 = \mathcal{Z}(\{0, 0\}) = \sum_{n=0}^{\infty} (n + 1)(n + 2) q^{n+1} = \frac{q(1 + q)}{(1 - q)^3}. \quad (3.5)$$

Next, let us consider the singlet partition function for N complex scalars in $d = 4$ (see [27, 8] and also [30, 31, 32] for relevant earlier work). One way to define it is to gauge the $U(N)$ symmetry and consider the coupling of N scalars Φ_r to $U(N)$ gauge field \mathcal{A}_m with strength \mathcal{F}_{mn}

$$L = D_m \Phi_r^* D^m \Phi_r + \frac{1}{4g^2} \text{tr} \mathcal{F}_{mn} \mathcal{F}^{mn}. \quad (3.6)$$

We shall understand the limit $g \rightarrow 0$ as restricting the path integral to pure-gauge fields \mathcal{A}_m . In the case of $S^1 \times S^3$ there is a non-trivial holonomy U of \mathcal{A}_0 along S^1 that cannot be gauged away. The remaining path integral will then be over Φ_r and U or phases α_r in

$$\mathcal{A}_0 = U^{-1} \partial_0 U, \quad U = \text{diag}(e^{i\alpha_1 \tau}, \dots, e^{i\alpha_N \tau}), \quad \tau = \beta^{-1} x_0 \in (0, 1). \quad (3.7)$$

We thus get the following expression for the singlet partition function of $U(N)$ scalars

$$\hat{Z} = \int \prod_{r=1}^N d\alpha_r e^{-\Gamma(\alpha; \beta)} , \quad (3.8)$$

$$\Gamma(\alpha; \beta) = \mu(\alpha) + \bar{F}(\alpha; \beta) , \quad \mu(\alpha) = -\frac{1}{2} \sum_{r \neq s=1}^N \ln \sin^2 \frac{\alpha_r - \alpha_s}{2} , \quad (3.9)$$

$$\bar{F} = \ln \det [-(\partial_0 + \mathcal{A}_0)^2 - \mathbf{D}^2 + 1] = \sum_{r=1}^N \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} d_n \ln \left[(2\pi k + \alpha_r)^2 \beta^{-2} + \omega_n^2 \right] . \quad (3.10)$$

Here $\mu(\alpha)$ is the contribution of the $U(N)$ invariant measure $[U^{-1}dU]$ and d_n and ω_n are as in (3.2),(3.5). To take the large N limit let us introduce the normalized $(\int_{-\pi}^{\pi} d\alpha \rho(\alpha) = 1)$ eigenvalue density $\rho(\alpha)$ and replace the integral over α_r by the path integral over the periodic field $\rho(\alpha)$ defined on a circle

$$\hat{Z} \Big|_{N \rightarrow \infty} = \int [d\rho] e^{-F(\rho; \beta)} , \quad (3.11)$$

$$F = N^2 \int d\alpha d\alpha' K(\alpha - \alpha') \rho(\alpha) \rho(\alpha') + 2N \int d\alpha \rho(\alpha) Q(\alpha; \beta) , \quad (3.12)$$

$$K(\alpha) = -\frac{1}{2} \ln (2 - 2 \cos \alpha) , \quad Q(\alpha; \beta) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} d_n \ln \left[(2\pi k + \alpha)^2 \beta^{-2} + \omega_n^2 \right] . \quad (3.13)$$

Here the N^2 term came from the measure term μ in (3.9). Isolating the Casimir energy part and rearranging the sum we get

$$Q(\alpha; \beta) = \beta E_c + \bar{Q} , \quad E_c = \frac{1}{2} \sum_{n=0}^{\infty} d_n \omega_n , \quad \bar{Q}(\alpha; \beta) = \sum_{m=1}^{\infty} c_m(\beta) \cos(m\alpha) , \quad (3.14)$$

$$c_m(\beta) = -\frac{1}{m} \mathcal{Z}_0(m\beta) , \quad \mathcal{Z}_0(\beta) = \sum_{n=0}^{\infty} d_n e^{-\beta \omega_n} . \quad (3.15)$$

Splitting the normalized periodic function $\rho(\alpha)$ into the constant and non-constant parts, $\rho(\alpha) = \frac{1}{2\pi} + \tilde{\rho}(\alpha)$, we can write (3.12) as (using that \bar{Q} in (3.14) does not couple to the constant part of ρ)

$$F = 2N\beta E_c + N^2 \int d\alpha d\alpha' K(\alpha - \alpha') \tilde{\rho}(\alpha) \tilde{\rho}(\alpha') + 2N \int d\alpha \tilde{\rho}(\alpha) \bar{Q}(\alpha; \beta) . \quad (3.16)$$

Integrating over $\tilde{\rho}$ (or, equivalently, evaluating the path integral at the large N saddle point) gives finally

$$\hat{\Gamma} = -\log \hat{Z} \Big|_{N \rightarrow \infty} = 2N\beta E_c + \hat{F}(\beta) , \quad (3.17)$$

$$\hat{F} = -\sum_{m=1}^{\infty} m [c_m(\beta)]^2 = -\sum_{m=1}^{\infty} \frac{1}{m} \hat{\mathcal{Z}}(m\beta) , \quad \hat{\mathcal{Z}}(\beta) = [\mathcal{Z}_0(\beta)]^2 . \quad (3.18)$$

Thus the one-particle partition function corresponding to the large N limit of the singlet-sector partition function is given by the square of the free scalar one in (3.5) [27]. Repeating this argument in the real $O(N)$ scalar case one finds a similar result [8], i.e.

$$\hat{\mathcal{Z}}_{\text{U(N)}}(\beta) = [\mathcal{Z}_0(\beta)]^2, \quad (3.19)$$

$$\hat{\mathcal{Z}}_{\text{O(N)}}(\beta) = \frac{1}{2}[\mathcal{Z}_0(\beta)]^2 + \frac{1}{2}\mathcal{Z}_0(2\beta). \quad (3.20)$$

These are exactly the expressions that appear in the l.h.s. parts of the character relations (2.6) and (2.7).

The leading Casimir energy term in (3.17) is the same as in (3.3) for $2N$ real scalars, while the non-trivial β -dependent part $\hat{F}(\beta)$ is of order N^0 rather than of order N when the singlet constraint is not imposed.

While the scalar one-particle partition function \mathcal{Z}_0 (3.15) counts all operators built out of one scalar and its derivatives modulo equations of motion, the singlet partition function $\hat{\mathcal{Z}}$ counts all bilinear spin s currents modulo the conservation condition. Equivalently, it is the total one-particle partition function for the theory of massless higher spin fields in thermal cover of AdS_5 [27, 8]. Similar results are found [8] in the spin $\frac{1}{2}$ case with $\mathcal{Z}_0 \rightarrow \mathcal{Z}_{\frac{1}{2}} = \frac{4q^{\frac{3}{2}}}{(1-q)^3}$ and the sign plus in (3.7) replaced by the sign minus.

Let us now repeat the same computation using the Maxwell action $S_1 = \frac{1}{4} \int d^4x \sqrt{g} F_{\mu\nu} F^{\mu\nu}$ instead of the scalar one. For one real Maxwell field we get the well-known expression for the curved-space partition function in the usual covariant Feynman gauge

$$Z_1 = \frac{\det(-D^2)}{[\det(-g_{\mu\nu}D^2 + R_{\mu\nu})]^{1/2}}, \quad (3.21)$$

Specializing to the $S^1 \times S^3$ case (splitting $A_\mu = (A_0, A_i)$ $i, j = 1, 2, 3$) we find [33]

$$Z_1 = \left[\frac{\det(-D^2)}{\det(-g_{ij}D^2 + R_{ij})} \right]^{1/2} = \frac{1}{[\det(-g_{ij}D^2 + R_{ij})_\perp]^{1/2}} = \frac{1}{[\det \mathcal{O}_{1\perp}]^{1/2}}, \quad (3.22)$$

where $\mathcal{O}_{1ij} = (-\partial_0^2 - \mathbf{D}^2 + 2)_{ij}$, and we split the 3-vector field operator into the transverse ($D^i A_{i,\perp} = 0$) and longitudinal parts. The same expression can be obtained directly by choosing the $D^i A_i = 0$ gauge in the path integral, where the action becomes

$$L_1 = \frac{1}{2} \partial_0 A_i \partial_0 A^i + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} D^i A_0 D_i A_0. \quad (3.23)$$

The contribution of the integral over A_0 cancels against the ghost or measure determinant giving back (3.22).

From the spectrum of transverse 3-vector Laplacian $(-\mathbf{D}^2)_{1\perp}$ on S^3 the spectrum of $\mathcal{O}_{1\perp}$ in (3.22) is found to be is

$$\lambda_{k,n} = w_k^2 + \omega_n^2, \quad \omega_n = n + 2, \quad d_n = 2(n+1)(n+3). \quad (3.24)$$

The resulting Z_1 has the form (3.3) where the one-particle partition function is thus equal to the character of spin 1 doubleton representation in (2.5) (cf. (3.22))

$$\mathcal{Z}_1 = Z(\{1, 0\}_c) = \sum_{n=0}^{\infty} d_n e^{-\beta \omega_n} = \frac{2q^2(3-q)}{(1-q)^3} . \quad (3.25)$$

Let us now consider N complex Maxwell vectors and impose the singlet constraint. One way to do this is to start with $U(N+1)$ YM theory and split the $U(N+1)$ field into the $U(N)$ one \mathcal{A}_m , N complex vectors A_m in the fundamental of $U(N)$ and a singlet. Then the YM action can be written as in (3.6)

$$L = \frac{1}{2} F_{r mn}^* F_r^{mn} + \frac{1}{4g^2} \text{tr} \mathcal{F}_{mn} \mathcal{F}^{mn} , \quad F_{mn}^r = \mathcal{D}_m A_n^r - \mathcal{D}_n A_m^r , \quad (3.26)$$

where $\mathcal{D}_m = \mathcal{D}_m(\mathcal{A})$, $r = 1, \dots, N$. We rescaled A_m by g and ignored the decoupled singlet. Taking the limit $g \rightarrow 0$ (for fixed N) understood as localizing the path integral over \mathcal{A}_m on $\mathcal{F}_{mn} = 0$ or pure gauge configurations we get then the direct analog of (3.6), (3.7), i.e. the following generalization of (3.23)

$$L_1 = \mathcal{D}_0 A_{ri}^* \mathcal{D}_0 A_r^i + \frac{1}{2} F_{rij}^* F_r^{ij} , \quad (3.27)$$

where \mathcal{D}_0 contains \mathcal{A}_0 in (3.7) (we ignored the trivial decoupled contribution of A_0). Integrating over A_i^r and U we then get again the relations (3.8), (3.9), (3.10) where now d_n and ω_n are given by (3.24). The remaining derivation of the large N limit of the singlet partition function is then literally the same as above in the scalar case.

As a result, we get exactly the same expressions for the singlet-sector partition function as in (3.19), (3.7) with \mathcal{Z}_0 replaced by \mathcal{Z}_1 , i.e.⁶

$$\hat{\mathcal{Z}}_{\text{U(N)}}(\beta) = [\mathcal{Z}_1(\beta)]^2 = \left[\frac{2q^2(3-q)}{(1-q)^3} \right]^2 , \quad (3.28)$$

$$\hat{\mathcal{Z}}_{\text{O(N)}}(\beta) = \frac{1}{2} [\mathcal{Z}_1(\beta)]^2 + \frac{1}{2} \mathcal{Z}_1(2\beta) = \frac{1}{2} \left[\frac{2q^2(3-q)}{(1-q)^3} \right]^2 + \frac{1}{2} \frac{2q^4(3-q^2)}{(1-q^2)^3} . \quad (3.29)$$

These expressions are indeed consistent with the AdS/CFT, i.e. with counting of conserved currents or counting of states in the dual type C theory in AdS_5 as follows from the comparison with the identities for the corresponding characters in (2.11) and (2.12).

4 Casimir energy

The Casimir energy is determined by the same spectrum as the one-particle partition function \mathcal{Z} in (3.3), (3.4) and they may be directly related as

$$E_c = \frac{1}{2} \sum_n d_n \omega_n = \frac{1}{2} \zeta_E(-1) , \quad \zeta_E(z) = \sum_n \frac{d_n}{\omega_n^z} = \frac{1}{\Gamma(z)} \int_0^\infty d\beta \beta^{z-1} \mathcal{Z}(e^{-\beta}) . \quad (4.1)$$

⁶Note that the resulting canonical partition function in (3.17) is different from the canonical single-trace partition function of one-loop $SU(N)$ YM theory on $S^1 \times S^3$ [30, 34, 31] counting single-trace operators built out of any number of fields, not just two. It is given at large N by $\mathcal{Z}_{\text{YM}} = - \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \ln[1 - \mathcal{Z}_1(k\beta)]$, where φ is Euler's totient function.

One can show that the Casimir energy vanishes if the partition function \mathcal{Z} obeys the property $\mathcal{Z}(q) = \mathcal{Z}(q^{-1})$ [8]. This property is true for the square of the scalar partition function in (3.19) (cf. (3.5)) but is not valid for the second term in (3.7) in the $O(N)$ case. Since the Casimir energy is given by the linear in β part of the log of the total partition function in (3.3) it then follows that E_c for the spectrum corresponding to the singlet partition function (3.7) is the same as for one real 4d scalar, i.e. in the scalar or type A case of AdS/CFT we have

$$\text{scalar :} \quad (\hat{E}_c)_{U(N)} = 0, \quad (\hat{E}_c)_{O(N)} = E_{c0} = E_c(\{0,0\}) = \frac{1}{240}. \quad (4.2)$$

In the vector case (3.28) there is no $q \rightarrow q^{-1}$ invariance already in the $U(N)$ case. Since the combination $[\mathcal{Z}_1(\beta)]^2 - 2\mathcal{Z}_1(\beta) = -\frac{4(3-q)(1-3q)q^2}{(1-q)^6}$ is symmetric under $q \rightarrow q^{-1}$ in this case the singlet-sector Casimir energy is same as of 2 real Maxwell 4d vectors. In the $O(N)$ case the contribution of the first term in (3.29) is half of the $U(N)$ case one and the second term gives the same E_{c1} contribution. Thus we find

$$\text{vector :} \quad (\hat{E}_c)_{U(N)} = (\hat{E}_c)_{O(N)} = 2E_{c1} = 2E_c(\{1,0\}_c) = 2 \times \frac{11}{120}. \quad (4.3)$$

The relation between the characters or one-particle partition functions in (2.13),(2.14) then implies the corresponding relations for the total Casimir energy of the AdS₅ theory: in non-minimal and minimal type C theory we then get, respectively,

$$2E_c(4;0,0) + E_c(4;1,0)_c + 2 \sum_{s=2}^{\infty} E_c(2+s; \frac{s}{2}, \frac{s}{2}) + \sum_{s=2}^{\infty} E_c(2+s; \frac{s+2}{2}, \frac{s-2}{2})_c = 2E_{c1}, \quad (4.4)$$

$$2E_c(4;0,0) + \sum_{s=2}^{\infty} E_c(2+s; \frac{s}{2}, \frac{s}{2}) + \sum_{s=2,4,\dots}^{\infty} E_c(2+s; \frac{s+2}{2}, \frac{s-2}{2})_c = 2E_{c1}. \quad (4.5)$$

Note that here one does not need to worry about regularization of the sums over spins provided one uses the ζ -function prescription to define E_c as in (4.1). Namely, one is first to compute the sum over spins of all partition functions for finite q or the sum of their Mellin transforms $\zeta_E(z)$ and then to continue the result to the required $z = -1$ value.⁷ In view of (2.11),(2.13) and (4.3) the relation (4.4) expresses the equality of E_c computed for the product of two spin 1 doubleton representations to twice E_c for the single doubleton representation, i.e.

$$E_c(\{1,0\}_c \otimes \{1,0\}_c) = 2E_c(\{1,0\}_c). \quad (4.6)$$

Remarkably, the relations (4.4) or (4.6) and (4.5) are true also for the boundary theory conformal anomaly coefficients a and c [25] (these correspond to partition functions for S^4 or Ricci flat space instead of $S^1 \times S^3$).

⁷This procedure is equivalent to using an exponential cutoff $\exp[-\epsilon(s + \frac{1}{2})]$ in the sum over s and dropping all terms that are singular in the $\epsilon \rightarrow 0$ limit, see [8].

To interpret (4.6) from the point of view of AdS/CFT we observe that the full boundary CFT contribution to the Casimir energy is simply proportional to N (cf. (3.17)). This should correspond to the classical type C theory contribution in AdS_5 . Then the duality requires the one-loop bulk contribution to E_c to vanish, but it does not according to (4.4),(4.5). To reconcile this with AdS/CFT duality we follow the suggestion made in the real scalar and fermion cases [6, 8] and conjecture that the coefficient in front of the classical bulk theory action should be shifted by an integer from its naive value N . Namely, let us assume that the classical actions of the non-minimal type C theory dual to complex $U(N)$ Maxwell theory and the minimal type C theory dual to real $O(N)$ Maxwell theory have the form

$$S_{\text{non-min } U(N)} = (2N - 2)(S_0 + \dots) , \quad S_{\text{min } O(N)} = (N - 2) S_0 . \quad (4.7)$$

Here S_0 stands for the common sector of the two type C theories (cf. (2.11),(2.12)) Then the factor of two difference between the leading large N terms in the two actions will be consistent with the fact that the boundary theory Casimir energy of N complex fields is the same as that of the $2N$ real fields. The equal negative subleading terms will be required to cancel the equal one-loop corrections (4.4),(4.5) of the quantum fluctuations of the bulk fields, in agreement with the absence of the subleading in N term in E_c in the boundary theory.

It is interesting to note that the structure of the two actions in (4.7) is consistent also with the fact that $U(N)$ theory should be equivalent to $O(2N)$ one as far as “non-singlet” properties like Casimir energy or partition function on a space with trivial holonomy (e.g., S^4) are concerned: one complex field should be the same as 2 real ones. The reason for the triviality of the bulk action in the $U(1)$ ($N = 1$) case implied by (4.7) remains to be clarified further.

The above discussion has a straightforward generalization to the case when the boundary theory is represented by a combination of vectors, fermions and scalars and, in particular, to the supersymmetric case. For example, the relation (4.6) holds also if one replaces the spin 1 doubleton by a combination $\{1, 0\}_c + n_{\frac{1}{2}}\{\frac{1}{2}, 0\}_c + n_0\{0, 0\}$ which represents a superdoubleton for special choices of $n_{\frac{1}{2}}$ and n_0 [25]. In the case of $\mathcal{N} = 4$ superdoubleton ($n_{\frac{1}{2}} = 4$, $n_0 = 6$) one gets the duality between the singlet sector of the theory of N copies of $\mathcal{N} = 4$ supersymmetric Maxwell multiplet and a special $\mathcal{N} = 4$ supersymmetric higher spin theory in AdS_5 generalizing maximally supersymmetric 5d gauged supergravity [16, 18, 35].

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